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Evaluating the Technical Provisions for Traditional Brazilian Annuity Plans: Continuous-Time Stochastic Approach Based on Solvency Principles

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This article presents an approach for evaluating the liabilities of traditional Brazilian annuity plans, using a continuous-time stochastic approach based on modern solvency principles. The technical provisions are obtained by means of conditional expectation, under a real-world measure and considering the peculiar characteristics of each plan and the financial guarantees and profit participations (bonus and dividend plans) embedded in the annuity plans. We assume that policyholder behavior is not optimal, but we also illustrate a calculation of provision assuming optimal policyholder behavior to show the differences between both assumptions. In this article all explicit provisions formulas are derived, and several relevant conclusions about the values of these provisions are discussed.

1. INTRODUCTION

Similar to other annuity markets, Brazilian annuity plans are classified as defined benefit, defined contribution, and unit-linked plans. The first two are more traditional, and today providers endeavor to avoid marketing these because of the risks involved in their operations. Despite this healthy attitude, insurers still have considerable liabilities in these products, because of past sales, and must correctly account for their provisions. These traditional annuity plans are commercialized by life insurers to the general public as individual personal plans and to employers as group personal pension plans. In the case of the latter, an employer may affiliate employees to a group personal plan through an individual contract, but the insurer manages the contribution and the administration of benefits. It is important to highlight that these annuity plans are not occupational pension plans, which, in Brazil, are governed by specific rules.

The National Council of Private Insurance (CNSP) is the system's deliberative body and responsible for the settlement of the Brazilian government's policy guidelines and directives for the annuity plans studied in this article. On the other hand, the Superintendence of Private Insurance (SUSEP) is an executive body of the politics delineated by the CNSP and is the insurance commissioner, responsible for the supervision and control of the insurance market in Brazil. It is a member of the International Association of Insurance Supervisors (IAIS) and therefore observes the guidelines of this association in the insurance supervision. Furthermore, the CNSP and SUSEP follow the solvency concepts of Solvency II[1] to improve the Brazilian rules.

In defined benefit plans, the value of the retirement income is established at the time of purchase of the annuity plan, and the premiums are calculated according to the equivalence principle using the first order basis (Møller and Steffensen 2006). By contrast, in the defined contribution plan, the value of the income is based on the amount accumulated in the policyholder account. This plan must include a guaranteed annuity with its rate being fixed in the contract. Furthermore, during the deferral period, there is a guaranteed minimum rate of return, which is fixed in the contract. Both guarantees are fixed in the contract according to

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[1]Directive 2009/138/EC of the European Parliament and the Council, November 25, 2009.

regulation by the CNSP. The guaranteed minimum rate of return was an important guarantee until the mid-1990s when there were high inflation rates in Brazil. In the early 2000s, the unit-linked plans were developed.

These two peculiar characteristics differentiate Brazilian defined contribution plans from other plans offered throughout the world, such as individual retirement accounts (IRAs) and 401(k) plans in the United States. One can observe that the Brazilian defined contribution plan is similar to the defined benefit plan, the main difference being that the latter has a guaranteed minimum rate of return derived from mortality rates and interest rates.

As a regulatory demand, companies must calculate two “kinds” of provisions: the contractual technical provision and the economic technical provision. The former arises from the first order basis, and the other provision arises from the real basis, where, under solvency valuation, the assumptions regarding interest and mortality rates have to be estimated using current information and realistic predictions. Because of this concept, only the prospective method is used to obtain this kind of provision, which we refer to as an economic provision.

The purpose of this article is to show the calculation of the liabilities of these traditional annuity plans, using a continuous-time stochastic approach, and considering the solvency concepts from the IAIS and Solvency II. To achieve this, we take into account the peculiar characteristics of each plan, embedded financial guarantees, and profit participation. The continuous approach in the provision calculation has been extensively examined in notable texts, for example, by Wolthuis and Hoem (1990), Norberg (1991), and Møller and Steffensen (2007). Recently Christiansen et al. (2014) rewrote Thiele’s equation assuming reserve-dependent benefits for multistate life insurance, given that this dependence is linear.

Under Solvency II, to obtain the economic technical provision, insurers must use market-consistent valuation. The technical provision is calculated as the best estimate plus the risk margin or the market value of the financial instruments that perfectly replicate the future cash flows associated with the insurance obligations. In Brazil there is no relevant longevity reinsurance market. In addition, no assets connected to the Brazilian mortality rate are available in the financial market. Finally, there is a strict regulation of insurers’ asset allocation policies by limiting the range of assets that insurance companies can buy to cover their commitments to policyholders and beneficiaries. So we can conclude that the annuity market is incomplete and insurers cannot apply optimal hedging strategies to perfectly replicate the future cash flows. Assuming an approach similar to Solvency II, one must obtain the best estimate and the risk margin to evaluate the technical provisions.

Moreover, in this article we assume that policyholder behavior is not optimal, as concluded by Neves et al. (2014), where these authors provided evidence supporting this assumption for the Brazilian annuity market. They concluded that the surrender rates depend on the gender and age, as their results showed that surrender rates are higher among women and younger policyholders. Furthermore, they found that surrender rates were also affected during economic crises. Therefore, they concluded that policyholder behavior is far from optimal. The same conclusion was made by De Giovanni (2010), who affirmed that Kuo et al. (2003) and Kim (2005) presented strong evidence to support such a conclusion. It is important to note that, because the Brazilian policyholder can withdraw the contractual provision during the deferral period, if the optimal behavior condition is assumed, the policyholder always would do it when the difference between contractual provision and best estimate is positive.

According to contractual conditions, in the case of surrender during the deferral period, the policyholder withdraws exactly the value of the contractual provision. Additionally, in the defined contribution plan, if the policyholder dies, his or her beneficiaries also have the right to receive the value of the contractual provision. Thus, considering the Cantelli Theorem (Cantelli 1914; Milbrodt and Stracke 1997), stochastic surrender rates are applied in the expression that calculates the economic provision. In the defined contribution plan, the mortality rates are also used during the deferral period to apply the actuarial discounts.

In this article we demonstrate the formulas to calculate this embedded financial guarantee, which is included in the economic provision. As an additional exercise, we consider an optimal policyholder behavior, such as shown by Ballotta and Haberman (2003, 2006), Boyle and Hardy (2003), and Biffis and Millossovich (2006). Under these conditions, we show the calculation of the surrender option, which must be used to obtain the value of the best estimate of the economic provision.

In some of the annuity plans there is profit participation, and because of this it is important to highlight the calculation of the provisions assuming the profit participation rules. The main characteristics of Brazilian profit participation are that its calculation is continuous and the transfer to the provision occurs at fixed times and that there are two kinds of plans: the bonus and dividend plan.

This study aims to contribute toward the understanding of insurers, supervisors, stakeholders, and researchers to help them understand the calculation of the technical provisions of annuity plans, using modern actuarial concepts for solvency valuation. Because of the characteristics of the plans, the best estimate of the economic provision is estimated by means of conditional expectation using a real-world measure. Additionally, we present comparisons among the values of the technical provisions developed in this article.

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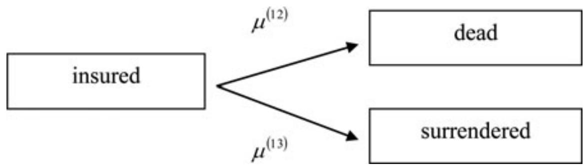

FIGURE 1. States for Annuity Plans during the Deferral Period, Where $\mu^{(12)}$ Is Force of Mortality and $\mu^{(13)}$ Is Force of Surrender.

The rest of the article is organized as follows. In the second section, the possible states for annuity plans are presented, and in the third section we demonstrate the formulas to calculate the provisions in the defined benefit plans. In the fourth section, we obtain the formulas for the calculation of the provisions in the defined contribution plans. The fifth section concludes the article.

2. MULTISTATE TRADITIONAL ANNUITY PLANS

Three possible states are found for annuity plans during the deferral (contributory) period: (1) insured, (2) dead, and (3) surrendered. In contrast, during the income payment period we find only two states: (1) retired and (2) dead. The transitions among the states are shown in Figures 1 and 2.

In general, in these annuity plans, we find two types of benefits: $c^{ij}(t)$, which is the value paid by the insurer on a transition from state i to state $j \neq i$ at time t, and $dB^i(t)$, which is the sojourn payment (minus premiums paid) in state i at time t, where $i, j, \ldots$ are the different states of the contract and $B^i(t)$ is the accumulated sojourn payment (minus premiums paid). In the defined contribution plan, the representation is clearer when we split $B^i(t)$ into two parts: $b^i(t)$, which is the accumulated incomes received in state i at time t, and $p^i(t)$, which is the accumulated premiums paid at time t.

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3. DEFINED BENEFIT PLAN PROVISIONS

In defined benefit plans, the value of the retirement income is established at the time of purchase of the annuity plan. The premiums are calculated according to the equivalence principle using the first order basis, considering the guaranteed retirement income. So the guaranteed minimum rate of return is derived from mortality rates and interest rates, both set in the annuity contract. The main difference between this plan and other defined benefit plans offered throughout the world is that the former is not an occupational plan and is sold to the general public. Furthermore, even when an employer affiliates employees to a personal group plan the insurer fully manages the annuity plan.

The benefit paid in the case of a transition from state 1 to state 2 before the time of retirement depends on the policy. The majority of policies do not pay benefits in case of death, but few policies refund to the beneficiaries a share or the totality of the premiums paid until the time of death. In case of surrender before the time of retirement, that is, transition from state 1 to state 3, the policyholder receives the contractual provision. Thus, the defined benefit plan has the following transition benefits:

1. $c^{12}(t)$ is an amount independent of the value of provisions or zero, being zero in the majority of defined benefit plans, and always being zero after the time of retirement and
2. $c^{13}(t) = V_c(t)$, where $V_c(t)$is the contractual technical provision at time t.

In traditional annuity plans, insurers must obtain two technical provisions: the contractual technical provision and the economic technical provision. The former is evaluated considering the contractual financial guarantee, and the latter is calculated in accordance with the solvency rules. According to Møller and Steffensen (2007), the contractual technical provision arises from the first order basis, which is a pair $(r^*, \mu^{*(12)})$ containing the first order interest rate and the first order mortality rate under which the guaranteed benefits are set according to the equivalence principle. These parameters are fixed in the policy at the time of purchase. In contrast, the economic technical provision arises from the real basis, which is the trio $(r, \mu^{(12)}, \mu^{(13)})$ containing the real interest rate, the real mortality rate, and the real surrender rate by which the economic technical provision accumulates. Under solvency valuation,

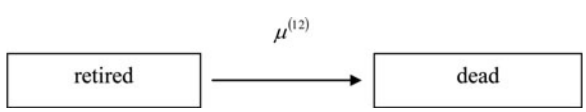

FIGURE 2. States of Annuity Plans during Income Payment Period, Where $\mu^{(12)}$ Is Force of Mortality.

such as the International Association of Insurance Supervisors (IAIS) through its Insurance Core Principles[2] and Solvency II, these real parameters must be estimated by means of current information and realistic predictions, using adequate, applicable, and relevant actuarial and statistical methods. So we do not work with retrospective calculations of the provisions and perform only the prospective method.

Some authors, such as Ballotta and Haberman (2003, 2006), Boyle and Hardy (2003), and Biffis and Millossovich (2006), have assumed optimal policyholder behavior in their pricing techniques. Nevertheless, in this article, we assume that policyholder behavior is not optimal, as concluded by De Giovanni (2010) and Neves et al. (2014). We consider that surrender rates have to be modeled through a stochastic model, such as have Tsai et al. (2002), Kuo et al. (2003), Kim (2005), Cox and Lin (2006), Loisel and Milhaud (2011), and Neves et al. (2014).

To measure the contractual provision it is not necessary to apply surrender rates, as can be seen in the following Thiele's differential equation. It occurs because of the Cantelli Theorem (Cantelli 1914; Milbrodt and Stracke 1997), since in the case of surrender, the insured receives exactly the value of the contractual provision:

$$dV_c(t) = V_c(t-)\,r^* dt - dB^1(t) - \mu^{*(12)}_{x+t}\left(c^{12}(t) - V_c(t)\right) dt, \tag{1}$$

where

$V_c(t)$ is the contractual technical provision at time t
$c^{12}(s) = 0, \forall s \geq n$
$r^*$and $\mu^{*(12)}_{x+u}$ are deterministic at $t \leq s \leq \infty$
$\mu^{*(12)}_{x+u}$ is the force of mortality at age $x + u$ fixed in the policy
r^* is the interest rate fixed in the policy
$dB^1(s)$ is the benefit minus premium paid in state 1 at time s
x is the present age of the policyholder and
$x + n$ is the retirement age.

Thus, the prospective contractual provision is denoted as the following formula (Eq. [2]) (Wolthuis and Hoem 1990). This provision is obtained as the unique solution of Equation (1), as demonstrated by Milbrodt and Helbig (1999) for payment functions that do not depend on the provision and by Christiansen et al. (2014) for reserve-dependent payments:

$$V_c(t) = \int_t^{\infty}\left[\exp\left(-\int_t^s r^* du - \int_t^s \mu^{*(12)}_{x+u} du\right)\left(dB^1(s) + \mu^{*(12)}_{x+s} c^{12}(s)\, ds\right)\right]. \tag{2}$$

To obtain the economic provision it is important to observe that, under solvency principles, insurers must use market-consistent valuations for assets and liabilities. Thus, liabilities should be valued at the amount for which they could be transferred or settled. Solvency II determines that the value of technical provisions shall be equal to the sum of a best estimate and a risk margin, where the last one must be such as to ensure that the technical provision is market-consistent. On the other hand, where future cash flows associated with insurance obligations can be replicated reliably using financial instruments for which a reliable market value is observable, the value of technical provisions associated with those future cash flows is determined based on the market value of these financial instruments. The economic technical provision, assuming market-consistent valuation, is obtained through the following process. The technical provision is defined in a filtered probability space $(\Omega, F, \{F_t\}_{t\geq 0}, Q)$ under the risk-neutral measure as follows:

$$V_{e,SII}(t) = E^Q\left(\int_t^{\infty}\left[\exp\left(-\int_t^s r_u du - \int_t^s \left(\mu^{(12)}_{x+u} + \mu^{(13)}_{x+u}\right) du\right)\left(dB^1(s) + \mu^{(12)}_{x+s} c^{12}(s)\, ds + \mu^{(13)}_{x+s} V_c(s)\, ds\right)\right] | F_t\right), \tag{3}$$

where

$V_c(s)$ is measurable and deterministic at $t \leq s \leq \infty$
$V_{e,SII}(t)$ is the economic technical provision at time t, under a neutral-risk measure

[2]Insurance Core Principles Standards, Guidance and Assessment Methodology, October 1, 2011. This publication is available on the IAIS website (www.iaisweb.org).

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r_u, $\mu_{x+u}^{(12)}$ and $\mu_{x+u}^{(13)}$ are stochastic and under a neutral-risk measure
F_t is a σ-algebra included in $F$and if $s \leq t,\ F_s \subset F_t$
$\mu_{x+s}^{(12)}$ is the force of mortality at age $x+s$
$\mu_{x+s}^{(13)}$ is the surrender rate at age $x+s$
$\mu_{x+u}^{(13)} = 0, \forall u \geq n$ and
r_u is the real interested rate at time u.

Like most annuities markets, the Brazilian annuity market is incomplete, and, since the insurer's asset allocation is subject to regulation constrains, insurers in practice cannot apply optimal hedging strategies (Gatzert and Kling 2007) to replicate their future cash flows. So, assuming market-consistent valuation, to evaluate the economic technical provision one must obtain a best estimate and a risk margin. The best estimate is measured by the probability-weighted average of future cash flows, taking into account the time value of money using the concepts adopted in Solvency II. In turn, the margin risk is the amount that ensures that the value of estimated provision is equivalent to the value that insurance would be expected to require in order to take over and meet the insurance obligations (EIOPA 2014). Because of the characteristics of the Brazilian market, we should not use the equivalent martingale measure (risk-neutral measure), and consequently, in this article, we assume that the best estimate of the economic provision is evaluated using a real-world measure (objective measure) under filtered probability space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ as follows:

$$V_{be}(t) = E\left(\int_t^{\infty}\left[\exp\left(-\int_t^s r_u du - \int_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right)\left(dB^1(s) + \mu_{x+s}^{(12)} c^{12}(s)\, ds + \mu_{x+s}^{(13)} V_c(s)\, ds\right)\right] | F_t\right), \tag{4}$$

where

$V_c(s)$ is measurable and deterministic at $t \leq s \leq \infty$
r_u,$\mu_{x+u}^{(12)}$ and $\mu_{x+u}^{(13)}$ are stochastic and under a real-world measure
$\mu_{x+s}^{(12)}$ is the force of mortality at age $x+s$
$\mu_{x+s}^{(13)}$ is the surrender rate at age $x+s$
r_u is the real interested rate at time$u$
$x+n$ is the retirement age
$\mu_{x+u}^{(13)} = 0, \forall u \geq n$ and
$c^{12}(s) = 0, \forall s \geq n$.

To evaluate the technical provision one must obtain the value of the risk margin. Solvency II assumes that the risk margin is the cost of providing an amount of eligible own funds equal to the solvency capital requirement necessary to support the plan obligations over the lifetime thereof (EIOPA 2014). So, following the expression (Eq. [5]) defined in EIOPA (2014), we can obtain the risk margin and consequently the economic provision. Nevertheless, one can note that the risk margin value depends on the solvency capital requirements rules:

$$R(t) = CoC(t) \sum_{s\geq 0} SCR(t+s) \big/ (1 + i_{t+s+1})^{s+1}, \tag{5}$$

$$V_e(t) = V_{be}(t) + R(t), \tag{6}$$

where

$V_e(t)$ is the economic technical provision at time t
$V_{be}(t)$ is the best estimate of the economic technical provision at time t
$R(t)$ is the risk margin at time t, being always positive
$CoC(t)$ is the cost-of-capital rate t
$SCR(t+s)$ is the solvency capital requirement to support the plan obligations at time $t+s$
i_{t+s} is the basic risk-free rate for maturity s at time t and
$s = 0, 1, 2, 3, \ldots$; this represents the discrete lifetime of the obligations.

As one can see in Equation (4), differently from determining the contractual provision, to measure the best estimate we must consider surrender rates during the deferral period.

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For asset-liability management as well as for estimating the market risk, one of the more relevant issues is the variation of the economic provision. As one has the right to withdraw the contractual provision, the volatility of the best estimate of the economic provision is not the same as the one presented by a hypothetical similar market-traded bond. When an investor sells its bond in the market, it receives back its economic value. In the case of a defined benefit plan, the investor will receive the contractual provision. Furthermore, in an economic solvency valuation, the flows of assets and liabilities will be different at the time of surrender. So the surrender flows must be considered in the asset-liability management. Consequently the surrender/lapse capital risk should be valued by insurers.

In these plans, the first order interest rates and mortality rates are contractually guaranteed as a minimum return. As a result, during the deferral period, a financial guarantee allows a policyholder to withdraw the value of contractual provision, even if this value is higher than the best estimate of the economic provision. To measure the best estimate of this financial guarantee, we must obtain a hypothetical value of the best estimate of the economic provision assuming that the policyholder can withdraw his or her own best estimate. So the financial guarantee is equal to the value of the best estimate of the economic provision (Eq. [4]) minus that hypothetical value, which is denoted in this article as a "pseudo-" best estimate and is obtained as follows:

$$V'_{be}(t) = E\left(\int_t^{\infty}\left[\exp\left(-\int_t^s r_u du - \int_t^s \left(\mu_{x+u}^{(12)}\right) du\right)\left(dB^1(s) + \mu_{x+s}^{(12)} c^{12}(s)\,ds\right)\right] |F_t\right). \tag{7}$$

Thus, the best estimate of the economic value of that embedded guarantee is

$$G_{be}(t) = \left(V_{be}(t) - V'_{be}(t)\right)^+. \tag{8}$$

Since $\mu_{x+u}^{(13)} = 0, \forall u \geq n$, one can observe that after the time of retirement $V_{be}(u) = V'_{be}(u)$. The embedded financial guarantee is obtained using the following formula:

$$G_{be}(t) = E\left(\begin{array}{l}\int_t^{\infty} \exp\left(-\int_t^s r_u du - \int_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right)\left(dB^1(s) + \mu_{x+s}^{(12)} c^{12}(s)\,ds + \mu_{x+s}^{(13)} V_c(s)\,ds\right) - \\ \exp\left(-\int_t^s r_u du - \int_t^s \left(\mu_{x+u}^{(12)}\right) du\right)\left(dB^1(s) + \mu_{x+s}^{(12)} c^{12}(s)\,ds\right) |F_t\end{array}\right)^+. \tag{9}$$

Consequently,

$$\begin{aligned}G_{be}(t) = E\Bigg(&\int_t^{\infty} \exp\left(-\int_t^s r_u du - \int_t^s \mu_{x+u}^{(12)} du\right) \\ &\times\left[\exp\left(-\int_t^s \mu_{x+u}^{(13)} du\right)\mu_{x+s}^{(13)} V_c(s)\,ds - \left(1 - \exp\left(-\int_t^s \mu_{x+u}^{(13)} du\right)\right)\left(dB^1(s) + \mu_{x+s}^{(12)} c^{12}(s)\,ds\right)\right] |F_t\Bigg)^+.\end{aligned} \tag{10}$$

From the embedded financial guarantee, we can present the following corollaries.

Corollary 1: *If* $V_c(s) \leq V_{be}(s) \quad \forall s \in [t, n) \Rightarrow V_{be}(t) \leq V'_{be}(t)$ *and* $G_{be}(t) = 0$.

Corollary 2: *If* $V_c(s) > V_{be}(s) \quad \forall s \in [t, n) \Rightarrow V_{be}(t) > V'_{be}(t)$ *and* $G_{be}(t) > 0$.

It is import to note that the risk margin is positive and depends on the capital solvency requirements rules. Therefore, even if $V_c(s) > V_{be}(s)$, we cannot affirm that $V_c(s) > V_e(s)$.

Proof. Corollaries 1 and 2 can be easily demonstrated using the Cantelli Theorem. Applying this theorem we observe that if $V_c(s) = V_{be}(s) \quad \forall s \in [t, n) \Rightarrow V_{be}(t) = V'_{be}(t)$. It happens because the withdrawal amount is always equal to the value of the best estimate. So, in this situation, the insurer does not have to apply surrender rates in Equation (4). Given this perception, it is easy to see, comparing Equations (4) and (7), that

if $V_c(s) < V_{be}(s) \quad \forall s \in [t, n) \Rightarrow V_{be}(t) < V'_{be}(t)$ and
if $V_c(s) > V_{be}(s) \quad \forall s \in [t, n) \Rightarrow V_{be}(t) > V'_{be}(t)$.

Furthermore, from Equation (10) we find Corollary 3.

Corollary 3: If $E\left(\int_t^\infty \exp\left(-\int_t^s r_u du - \int_t^s \mu_{x+u}^{(12)} du - \int_t^s \mu_{x+u}^{(13)} du\right) \mu_{x+s}^{(13)} V_c(s)\, ds | F_t\right)$

$$> E\left(\int_t^\infty \exp\left(-\int_t^s r_u du - \int_t^s \mu_{x+u}^{(12)} du\right)\left(1 - \exp\left(-\int_t^s \mu_{x+u}^{(13)} du\right)\right)\left(dB^1(s) + \mu_{x+s}^{(12)} c^{12}(s)\, ds\right) | F_t\right) \Rightarrow G_{be}(t) > 0.$$

Corollary 3 is a development of Equation (10), and it ensures that, if the present value of the surrender lump sums is higher than the present value of the benefits, the embedded financial guarantee that allows a policyholder to withdraw the value of contractual provision has a value. This value is automatically included in the value of the best estimate of the economic provision.

Additionally, as an exercise, we could assume optimal policyholder behavior. In this situation, it is important to price the surrender option to obtain the value of economic provision. As the policyholder can withdraw the contractual provision during the deferral period, in the optimal behavior condition, the policyholder will do it when the difference between contractual provision and best estimate is positive. This is similar to an American put option with a strike (exercise) price equal to the contractual provision. Given that the policyholder would withdraw the provision if, and only if, his or her contractual provision is higher than the best estimate, the surrender rates are not used in the calculation of the provisions, being the best estimate evaluated using a real-world measure under filtered probability space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$ as in the following formula. The economic provision and the risk margin are obtained using Equations (5) and (6):

$$V_{be,opt}(t) = E\left(\int_t^\infty \left[\exp\left(-\int_t^s r_u du - \int_t^s \mu_{x+u}^{(12)} du\right)\left(dB^1(s) + \mu_{x+s}^{(12)} c^{12}(s)\, ds\right)\right] | F_t\right) + O(t), \tag{11}$$

where

$$O(t) = \max\left(\max_{t\leq k<n}\left\{E\left(\exp\left(-\int_t^k r_u du - \int_t^k \mu_{x+u}^{(12)} du\right)\left(V_c(k) - V_{be,opt}^*(k)\right)\right) | F_t\right\}; 0\right), \tag{12}$$

$$V_{be,opt}^*(k) = E\left(\int_k^\infty \left[\exp\left(-\int_k^s r_u du - \int_k^s \mu_{x+u}^{(12)} du\right)\left(dB^1(s) + \mu_{x+s}^{(12)} c^{12}(s)\, ds\right)\right] | F_t\right), \tag{13}$$

where

$V_{be,opt}(t)$ is the best estimate under optimal policyholder behavior at time t
$V_{be,opt}^*(t)$ is the expected value of the insurance obligations at time t
$O(t)$ is the value of surrender option at time t and
$x + n$ is the retirement age.

We obtain some conclusions in relation to this provision:

- $V_{be,opt}(u) \geq V_c(u)\ \forall u \in [t, n)$
- Whenever the maximum value of the option is obtained at time u, then $V_{be,opt}(u) = V_c(u)$
- $V_{be,opt}(u) > V_{be}(u)\ \forall u \in [t, n)$ and
- $V_{be,opt}(u) = V_{be}(u)\ \forall u \geq n$.

Corollary 4: *Assuming optimal policyholder behavior under solvency principles, the option to surrender must be accounted for, but it is never exercised in this annuity plan.*

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Proof. From the analysis of Equations (11) and (12) it is easy to note that $V_{be,opt}(u) \geq V_c(u) \, \forall u \in [t, n)$. Then the policyholder never exercises the surrender option given the assumption of optimal behavior under solvency principles.

3.1. Profit Participation

In several defined benefit plans, profit participation is established in their contracts, with surplus dividends depending on the accumulated contractual provision. The rules for paying the profit to the policyholder are established in the contract. The most important characteristic of the annuity plans studied is that the calculation of the profit is continuous and the transfer to the policyholders occurs at predefined times, such as semiannually, annually, and biannually. There are two kinds of plans: a bonus plan, where the dividends are paid immediately in cash to policyholder, and a dividend plan, where the dividends are deposited in the technical provision to increase future incomes.

3.1.1. Bonus Plan

In a bonus plan, if the value of accumulated profit is higher than zero at the time of payment, then this amount is paid immediately in cash to the policyholder. The percentage of the surplus paid to the policyholder is determined in the contract. At the time of payment, when the value of accumulated profit is negative, the value returns to zero. In this kind of plan, the contractual provision does not change. Nonetheless, assuming solvency principles, the cash out-flows to obtain the best estimate should include the forecasted profits, considering that the future commitments of the insurer with the policyholder are subject to change. This decision is in line with the definition of best estimate of the economic provision, which is presented above.

The future profit estimation at time t is found by using the following formula (Norberg 2001; Ramlau-Hansen 1991), which follows exactly the rule established in the studied annuity plan:

$$p(t) = V_c(t-)\,\beta\left(r_t - r^* + \mu_{x+t}^{(12)} - \mu_{x+t}^{*(12)}\right), \tag{14}$$

where

$p(t)$ is the profit at time t
$V_c(t)$ is measurable and deterministic at t and
β is the percentage of the surplus paid to the policyholder define in the contract.

Because of the characteristics of the contract, the transfer of the profit to the policyholder occurs at the end of the profit calculation time. So, if it is positive, the amount is paid to the policyholder immediately. As a result, we must forecast the profit payment at the defined the periodicity as follows:

$$dP(vk) = \left(E\left(\int_{((v-1)k,vk]} \exp\left(\int_s^{vk} r_u du\right) p(s)\,ds | F_t\right)\right)^+, \tag{15}$$

where

$dP(vk)$ is the future profit payment at time vk
k is the periodicity, being $k = 1$ if annual, $k = 0.5$ if biannual, ... and
$v \in \aleph - \{0\}$, $v = 1, 2, 3, 4, \ldots$.

So, assuming the payment of future profit as policyholder benefits and that they must be presumed as cash out-flows, the best estimate of the economic provision under solvency principles is obtained by revising Equation (4) as follows:

$$\begin{aligned} V_{be,bp}(t) = \mathrm{E}\Bigg(\int_t^\infty \Bigg[\exp\left(-\int_t^s r_u du - \int_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right) \\ \times \left(dB^1(s) + I_{(s=vk)} \cdot dP(s) + \mu_{x+s}^{(12)} c^{12}(s)\,ds + \mu_{x+s}^{(13)} V_c(s)\,ds\right)\Bigg] | F_t\Bigg), \end{aligned} \tag{16}$$

where

$V_{be,bp}(t)$ is the best estimate of the economic technical provision of a bonus plan at time t and
$I_{(s=vk)}$ is an indicator function assuming unit value if $s = vk$ and zero otherwise.

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From Equation (16), one can deduce the next corollary.

Corollary 5: $V_{be,bp}(t) > V_{be}(t)$ *if* $\exists\, dP(s) > 0\ \forall s \in [t, \infty]$

If there is an estimate of a future profit payment the insurer's commitments increase, then the best estimate of a bonus plan is greater than a plan without profit participation. When we analyze the risk margin we reach the same conclusion, considering that the margin risk is calculated using Equation (5). Thus, the economic provision of the bonus plan is greater than a plan without profit participation if a future profit payment is estimated.

Furthermore, we assume hypothetical conditions to perform a sensitivity analysis of the relationship between the first order parameters and real parameters. First, we can find two corollaries in relation to the provisions, giving different interest rates and hypothetically assuming that the mortality rates are equal (i.e., $\mu^* = \mu$).

Corollary 6: *Whenever* $r_s > r^* \forall s \in [t, \infty]$ *and* $0 < \beta < 1 \Rightarrow V_c(t) > V_{be,bp}(t) > V_{be}(t)$.

In this situation there will be profit payments, so $V_{be,bp}(t) > V_{be}(t)$. But if the percentage of the surplus (β) is less than a unit, then $V_c(t) > V_{be,bp}(t)$. We can highlight that since the risk margin is positive and depends on the capital solvency requirements rules, we cannot affirm that $V_c(s) > V_{e,bp}(s)$ even if $V_c(s) > V_{be,bp}(s)$.

Corollary 7: *Whenever* $r^* > r_s \forall s \in [t, \infty]$ *and* $0 \leq \beta \leq 1 \Rightarrow V_c(t) < V_{be,bp}(t) = V_{be}(t)$.

In this situation, because of there not being profit payments, we know that $V_{be,bp}(t) = V_{be}(t)$. Furthermore, as the first order parameter's is higher, it is easy to conclude that $V_c(t) < V_{be,bp}(t)$.

Complementarily, we can present more two corollaries, giving different mortality rates and hypothetically assuming that interest rates are equal (i.e., $r^* = r$). The situation can happen if the mortality rates fixed in the contract are less or greater than estimated mortality rates.

Corollary 8: *Whenever* $\mu_{x+s}^{(12)} > \mu_{x+s}^{*(12)} \forall s \in [t, \infty]$ *and* $0 < \beta < 1 \Rightarrow V_c(t) > V_{be,bp}(t) > V_{be}(t)$.

Corollary 9: *Whenever* $\mu_{x+s}^{*(12)} > \mu_{x+s}^{(12)} \quad \forall s \in [t, \infty]$ *and* $0 \leq \beta \leq 1 \Rightarrow V_c(t) < V_{be,bp}(t) = V_{be}(t)$.

The interpretations of these corollaries are similar to those of the Corollaries 6 and 7.

3.1.2. Dividend Plan

The main difference between the dividend plan and the bonus plan is that if the value of a profit payment is higher than zero at the time of payment, this amount is deposited in the technical provision to increase future incomes. Each profit is used as a single premium to increase the value of future incomes. Thus, the contractual provision depends not only on the first order parameters but also on the profits deposited until the moment of the provision's calculation.

To calculate the contractual provision as well as the best estimate, it is necessary to know the values of each of the past profit payments, which are obtained from the following formula (Norberg 2001; Ramlau-Hansen 1991), considering the contractual characteristics:

$$dP^o(vk) = \left(\int_{((v-1)k, vk]} \exp\left(\int_s^{vk} r_u^o du \right) V_{c,dp}(s-)\, \beta \left(r_s^o - r^* + \mu_{x+s}^{o(12)} - \mu_{x+t}^{*(12)} \right) ds \right)^+ , \tag{17}$$

where

$dP^o(vk)$ is the profit paid at time vk
$vk < t$
t is provision evaluation time
$V_{c,dp}(t)$ is the contractual technical provision of a dividend plan at time t
r_s^o is the interest rate observed at time s and
$\mu_{x+s}^{o(12)}$ is the force of mortality observed at time s
k is the periodicity, being $k = 1$ if annual, $k = 0.5$ if biannual, ... and
$v \in \aleph - \{0\}$, $v = 1, 2, 3, 4, \ldots$.

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In this kind of plan, each profit $\left(dP^o\right)$ generates an additional value of income, as if it were an insurance premium. So the units of each additional income purchased before the time of provision evaluation are calculated dividing the profit by the contractual annuity, which is fixed in the policy (Eq. [19]):

$$k^o(s) = I_{(s=vk)} \frac{dP^o(s)}{a(s)}, \tag{18}$$

where

$vk < t$
$a(s)$ is the contractual annuity:

$$\begin{cases} a(s) = \int_n^\infty \exp\left(-\int_s^k r^* dk - \int_s^k \mu_{x+k}^{*(12)} dk\right) du, & \text{if } s < n \\ a(s) = \int_s^\infty \exp\left(-\int_s^k r^* dk - \int_s^k \mu_{x+k}^{*(12)} dk\right) du, & \text{if } s \geq n \end{cases} \tag{19}$$

$k^o(s)$ is the additional income purchased at time s and
n is the time of retirement.

The amount of additional income accumulated until the time of evaluation is calculated using the following expression:

$$dK^o(u) = \int_{(0,t)} k^o(s) ds, \; s < t \text{ and } u \geq n, \tag{20}$$

where $dK^o(u)$ is the additional income that is paid at time u, which was acquired by profits paid before the time of evaluation.

So, as presented in Ramlau-Hansen (1991), the units of additional income bought by past profits must be added to the first order basis provision. Thus, as there is a contractual obligation to pay additional income sourced from past profits, Equation (2) must be revised as follows:

$$V_{c,dp}(t) = \int_t^\infty \left[\exp\left(-\int_t^s r^* du - \int_t^s \mu_{x+u}^{*(12)} du\right)\left(dB(s) + dK^o(s) + \mu_{x+s}^{*(12)} c^{12}(s) ds\right)\right], \tag{21}$$

where $V_{c,dp}(t)$ is the contractual technical provision of a dividend plan at time t.

In contrast, to obtain the best estimate of the economic provision, considering solvency principles, one also needs to take into account the units of additional income that will be obtained by each future profit payment (Eq. [15]). These cash out-flows must be calculated to evaluate the best estimate under solvency principles. So units of additional income are obtained by dividing the results of Equation (15) and Equation (19):

$$k(s) = I_{(s=vk)} \frac{dP(s)}{a(s)}, \quad s \geq t, \tag{22}$$

where $k(s)$ is the estimate of the additional income purchased at time s.

In addition, the accumulated amount of future units is found as

$$dK(u) = \int_t^u k(s) ds, \quad u \geq n, \tag{23}$$

where $dK(u)$ is the estimate of additional income that will be paid at time u.

As a result, the best estimate of the economic provision is obtained by the following formula:

$$V_{be,dp}(t) = \mathrm{E}\left(\int_t^{\infty}\left[\begin{array}{l}\exp\left(-\int_t^s r_u du - \int_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right) \\ \left(dB^1(s) + dK^o(s) + dK(s) + \mu_{x+s}^{(12)} c^{12}(s)\, ds + \mu_{x+s}^{(13)} V_c(s)\, ds\right)\end{array}\right] |F_t\right), \tag{24}$$

where $V_{be,dp}(t)$ is the best estimate of the economic technical provision of a dividend plan at time t. To calculate the risk margin and economic provision one has to follow Equations (5) and (6).It is important to highlight some conclusions about this kind of profit participation.

Corollary 10: $V_{c,dp}(t) > V_c(t)$ if $\exists\, dP^0(u) > 0\ \forall u < t$.

Thus, if there has been profit paid in the dividend plan, the additional income $\left(dK^o\right)$ will be greater than zero and the contractual obligations with the policyholder will also become larger (see Eq. [21]). So a dividend plan tends to produce a larger contractual provision than a bonus plan and a plan without profit participation, considering the additional income accumulated. One can see it comparing Equations (2) and (21).

Corollary 11: $V_{be,dp}(t) > V_{be}(t)$ *if* $\exists\, dP^o(u) > 0\ \forall u < t$ *or* $\exists\, dP(s) > 0\ \forall s \in [t, \infty]$.

If there has been a past profit payment, there will be additional incomes $\left(dK^o\right)$, and if we forecast a future profit payment, it will also generate other additional income (dK). It can be noted by comparing Equations (4) and (24) that both additional incomes increase the value of the best estimate. Then the economic value of the insurer's commitments in the dividend plan tends to be higher than a plan without profit participation, considering the additional incomes. One can see it comparing Equations (4) and (24).

We also compared the economic provisions of both profit participation plans. There is a tendency for the best estimate as well as the economic provision to be higher in the dividend plan. We can see this by comparing Equations (16) and (24). The latter has the increment of additional incomes sourced from the past profit payments. Furthermore, the forecasted profits are computed differently in the both plans. In the dividend plan, the profit payments are converted to units of income using the first order basis, but the best estimate of these commitments depends on the economic value of the annuity. This is calculated as follows:

$$\begin{cases} a_e(s) = \int_n^{\infty} \exp\left(-\int_s^k r dk - \int_s^k \mu_{x+k}^{(12)} dk\right) du, & \text{if } s < n \\ a_e(s) = \int_s^{\infty} \exp\left(-\int_s^k r dk - \int_s^k \mu_{x+k}^{(12)} dk\right) du, & \text{if } s \geq n. \end{cases} \tag{25}$$

In general, in Brazil, the contractual annuity of these traditional plans is underpriced, so

$$\begin{aligned} &\frac{dP(s)}{a(s)} a_e(s) > dP(s) \quad \forall s \geq t \\ &\Rightarrow \mathrm{E}\left(\int_t^{\infty}\left[\exp\left(-\int_t^s r_u du - \int_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right)\left(I_{(s=vk)}.dP(s)\right)\right] |F_t\right) \\ &< \mathrm{E}\left(\int_t^{\infty}\left[\exp\left(-\int_t^s r_u du - \int_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right) dK(s)\right] F_t\right). \end{aligned} \tag{26}$$

As a result, there is a propensity for the best estimate to be higher in the dividend plan.

Furthermore, we also assume conditions for performing a sensitivity analysis of the relationship between the first order parameters and real parameters. Initially, we suppose different interest rates and hypothetically assume that the mortality rates are equal (i.e., $\mu^* = \mu$).

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Corollary 12: *Whenever* $r_s > r^* \; \forall s \in [t, \infty]$ *and* $0 < \beta < 1 \Rightarrow V_{c,dp}(t) > V_{be,dp}(t) > V_{be}(t)$.

In this situation, there will be additional income (dK) generated by future profit payments, so $V_{be,dp}(t) > V_{be}(t)$. But if the percentage of the surplus (β) is less than a unit, then $V_{c,dp}(t) > V_{be,dp}(t)$.

Corollary 13: *Whenever* $r^* > r_s \; \forall s \in [t, \infty]$ *and* $0 \leq \beta \leq 1 \Rightarrow V_{c,dp}(t) < V_{be,dp}(t)$.

In this defined condition we do not have the additional incomedK. But, as the first order parameter is higher, it is easy to conclude that $V_{c,dp}(t) < V_{be,dp}(t)$.

Complementarily, we can show two more corollaries, giving different mortality rates and hypothetically assuming that interest rates are equal (i.e., $r^* = r$).

Corollary 14: *Whenever* $\mu_{x+s}^{(12)} > \mu_{x+s}^{*(12)} \; \forall s \in [t, \infty]$ *and* $0 < \beta < 1 \Rightarrow V_{c,dp}(t) > V_{e,dp}(t) > V_e(t)$.

Corollary 15: *Whenever* $\mu_{x+s}^{*(12)} > \mu_{x+s}^{(12)} \; \forall s \in [t, \infty]$ *and* $0 \leq \beta \leq 1 \Rightarrow V_{c,dp}(t) < V_{e,dp}(t)$.

The interpretations of these corollaries are similar to Corollaries 12 and 13.

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4. DEFINED CONTRIBUTION PLAN PROVISIONS

In this type of plan, the value of the income is based on the amount accumulated in the policyholder account and the guaranteed annuity rate fixed in the contract. Additionally, during the deferral period, the insurer guarantees a minimum rate of return predetermined in the contract. The evaluation of the provisions is similar to that presented in Section 3, but in this plan, during the contribution period, the guaranteed minimum rate of return is a fixed rate of interest.

In the case of death of the policyholder before the date of retirement, the benefit paid is the contractual provision. The same benefit is paid when the policyholder surrenders before the time of retirement; both come from the regulatory framework. In consequence, the defined contribution plan has the following transition benefits: $c^{12}(t) = c^{13}(t) = V_c(t)$.

As a function of the financial guarantee in the deferral period, Equation (1) must be changed. By means of Thiele's differential equation below, one can note that the development of the contractual provision is different from that equation:

$$\begin{cases} dV_c^{dc}(t) = V_c^{dc}(t-)\, r^* dt + dp^1(t) & \text{if} \quad t < n \\ dV_c^{dc}(t) = V_c^{dc}(t-)\, r^* dt - db^1(t) + \mu_{x+t}^{*(12)} V_c^{dc}(t)\, dt & \text{if} \quad t \geq n \end{cases}, \tag{27}$$

where

$V_c^{dc}(t)$ is the contractual technical provision of the defined contribution plan at time t
r^* and $\mu_{x+u}^{*(12)}$ are deterministic at $t \leq s \leq \infty$
$\mu_{x+u}^{*(12)}$ is the force of mortality at age $x + u$ fixed on the policy
r^* is the interested rate fixed on the policy
$db^1(s)$ is the benefit paid in state (1) at time s
$dp^1(s)$ is the premium paid in state (1) at time s
x is the present age of the policyholder and
$x + n$ is the retirement age.

Thus, the prospective contractual provision is denoted as the following formula (Wolthuis and Hoem 1990):

$$\begin{cases} V_c^{dc}(t) = \int\limits_{[t,n)} \exp\left(-\int\limits_t^s r^* du\right)\left(-dp^1(s)\right) + \int\limits_n^\infty \exp\left(-\int\limits_t^s r^* du - \int\limits_n^s \mu_{x+u}^{*(12)} du\right) db^1(s) & \text{if } t < n \\ V_c^{dc}(t) = \int\limits_t^\infty \exp\left(-\int\limits_t^s r^* du - \int\limits_t^s \mu_{x+u}^{*(12)} du\right) db^1(s) & \text{if } t \geq n \end{cases}, \tag{28}$$

where

$db^1(s) = 0\ \forall s < n$
$dp^1(s) = 0\ \forall s \geq n$ and
$x + n$ is the retirement age.

In contrast, we obtain the best estimate of the economic technical provision of the defined contribution plan, assuming that policyholder behavior is not optimal and using a real-world measure under filtered probability space $(\Omega, F, \{F_t\}_{t\geq 0}, P)$, as follows:

$$V_{be}^{dc}(t) = E\left(\begin{array}{l}\left[\int\limits_{[t,n)} \exp\left(-\int\limits_t^s r_u du - \int\limits_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right)\left(-dp^1(s) + \mu_{x+s}^{(12)} V_c^{dc}(s) ds + \mu_{x+s}^{(13)} V_c^{dc}(s) ds\right)\right] + \\ \left[\int\limits_n^\infty \exp\left(-\int\limits_t^s r_u du - \int\limits_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right) db^1(s)\right] | F_t\end{array}\right), \quad (29)$$

where

$V_{be}^{dc}(t)$ is the best estimate of the economic technical provision of the defined contribution plan at time t
$V_c^{dc}(s)$ is measurable and deterministic at $t \leq s \leq \infty$
r_u, $\mu_{x+u}^{(12)}$ and $\mu_{x+u}^{(13)}$ are stochastic and under a real-world measure
$\mu_{x+s}^{(12)}$ is the force of mortality at age $x + s$
$\mu_{x+s}^{(13)}$ is the surrender rate at age $x + s$
$\mu_{x+u}^{(13)} = 0, \forall u \geq n$ and
r_u is the real interested rate at time u.

The risk margin and the economic provision are also evaluated using Equations (5) and (6). As one can see in Equation (29), differently from that of the contractual provision, to measure the best estimate of the economic provision one must consider surrender and mortality rates during the deferral period, because in case of death or surrender the withdrawn value is different to that of the best estimate. Thus, for asset-liability management as well for estimating the market risk, the mortality and surrender flows must be considered in the asset-liability management, resulting in the same problem discussed in Section 3.

The financial guarantee established in the contract allows a policyholder to withdraw the value of the contractual provision during the deferral period, and even when this value is higher than the best estimate, the same guarantee is offered to the beneficiaries in case of the policyholder's death. As in Section 3, to obtain the best estimate of this financial guarantee we evaluate the "pseudo-" best estimate's formula. In the following expression, we assume that in case of surrender or death the best estimate of the economic provision is withdrawn:

$$V_{be}^{dc'}(t) = E\left(\int\limits_{[t,n)} \exp\left(-\int\limits_t^s r_u du\right)\left(-dp^1(s)\right) + \int\limits_n^\infty \exp\left(-\int\limits_t^s r_u du - \int\limits_n^s \mu_{x+u}^{(12)} du\right) db^1(s) | F_t\right), \quad (30)$$

where $V_{be}^{dc}(u) = V_{be}^{dc'}(u)\ \forall u \geq n$ given that $\mu_{x+u}^{(13)} = 0\ \forall u \geq n$.

Using Equation (8), we have the following expression to obtain the financial guarantee:

$$G_{be}^{dc}(t) = E\left(\begin{array}{l}\left(\int\limits_{[t,n)} \exp\left(-\int\limits_t^s r_u du - \int\limits_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right)\left(-dp^1(s) + \mu_{x+s}^{(12)} V_c^{dc}(s) ds + \mu_{x+s}^{(13)} V_c^{dc}(s) ds\right)\right. \\ \left. \quad - \int\limits_{[t,n)} \exp\left(-\int\limits_t^s r_u du\right)\left(-dp^1(s)\right)\right) + \\ \left(\int\limits_n^\infty \exp\left(-\int\limits_t^s r_u du - \int\limits_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right) db^1(s) - \int\limits_n^\infty \exp\left(-\int\limits_t^s r_u du - \int\limits_n^s \mu_{x+u}^{(12)} du\right) db^1(s)\right) | F_t\end{array}\right)^+. \quad (31)$$

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Accordingly,

$$G_{be}^{dc}(t) = E\left(\begin{array}{l}\left[\int\limits_{[t,n)} \exp\left(-\int\limits_t^s r_u du - \int\limits_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right)\left(\mu_{x+s}^{(12)} V_c^{dc}(s)\, ds + \mu_{x+s}^{(13)} V_c^{dc}(s)\, ds\right)\right] + \\ \left[\int\limits_{[t,n)} \exp\left(-\int\limits_t^s r_u du\right)\left(dp^1(s)\right)\left(1 - \exp\left(-\int\limits_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right)\right)\right] - \\ \left[\int\limits_n^{\infty} \exp\left(-\int\limits_t^s r_u du - \int\limits_n^s \mu_{x+u}^{(12)} du\right)\left(db^1(s)\right)\left(1 - \exp\left(-\int\limits_t^n \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right)\right)\right] | F_t\end{array}\right)^+ . \quad (32)$$

Corollary 16: *If*

$$E\left(\begin{array}{l}\left[\int\limits_{[t,n)} \exp\left(-\int\limits_t^s r_u du - \int\limits_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right)\left(\mu_{x+s}^{(12)} V_c^{dc}(s)\, ds + \mu_{x+s}^{(13)} V_c^{dc}(s)\, ds\right)\right] + \\ \left[\int\limits_{[t,n)} \exp\left(-\int\limits_t^s r_u du\right)\left(dp^1(s)\right)\left(1 - \exp\left(-\int\limits_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right)\right)\right] | F_t\end{array}\right)$$

$$> E\left(\begin{array}{l}\left[\int\limits_{[t,n)} \exp\left(-\int\limits_t^s r_u du - \int\limits_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right)\left(\mu_{x+s}^{(12)} V_c^{dc}(s)\, ds + \mu_{x+s}^{(13)} V_c^{dc}(s)\, ds\right)\right] + \\ \left[\int\limits_{[t,n)} \exp\left(-\int\limits_t^s r_u du\right)\left(dp^1(s)\right)\left(1 - \exp\left(-\int\limits_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right)\right)\right] | F_t\end{array}\right) \Rightarrow G_{be}^{dc}(t) > 0.$$

From Equation (32) it is easy to note that the embedded guarantee has a value if the present value of surrender and death lump sums plus the value of the premiums that the policyholder would pay if he or she remained in the plan is higher than the present value of the future benefits that he or she would receive if not surrendered or died. This value is automatically included in the value of the economic provision.

As in the defined benefit plans, from the embedded financial guarantee, we can present the following corollaries.

Corollary 17: *If* $V_c^{dc}(s) \leq V_{be}^{dc}(s) \quad \forall s \in [t, n) \Rightarrow V_{be}^{dc}(t) \leq V_{be}^{dc'}(t)$ *and* $G_{be}^{dc}(t) = 0.$

Corollary 18: *If* $V_c^{dc}(s) > V_{be}^{dc}(s) \quad \forall s \in [t, n) \Rightarrow V_{be}^{dc}(t) > V_{be}^{dc'}(t)$ *and* $G_{be}^{dc}(t) > 0.$

As the risk margin is positive and depends on the capital solvency requirements rules, even if $V_c^{dc}(s) > V_{be}^{dc}(s)$ we cannot state that $V_c^{dc}(s) > V_e^{dc}(s)$. The proofs of the Corollaries 17 and 18 are equal to Corollaries 1 and 2.

4.1. Profit Participation

As in Section 3, in the defined contribution plans we also find two types of profit participation: the bonus plan and the dividend plan. The calculation of the profit is also continuous and with the transfer to the provision occurring at fixed times. The way these plans work is similar to that shown in Section 3.1, but now the plan does not guarantee mortality rates in the deferral period.

4.1.1. Bonus Plan

The contractual conditions are similar to those in Section 3.1.1. Nevertheless, because of the characteristics of the defined contribution plan, two ways are used to estimate the future profit at time t, the first one during the deferral period and the other

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after this time as follows:

$$\begin{cases} p^{dc}(t) = V_c^{dc}(t-)\,\beta\,(r_t - r^*)\,ds & \text{if } t < n \\ p^{dc}(t) = V_c^{dc}(t-)\,\beta\left(r_t - r^* + \mu_{x+t}^{(12)} - \mu_{x+t}^{*(12)}\right) ds & \text{if } t \geq n \end{cases}, \tag{33}$$

where $p^{dc}(t)$ is the profit at time $t$in a defined contribution plan.

Because the contract states that the profit transfers occur at the end of the profit calculation time, we forecast the profit payment at the defined periodicity as follows:

$$dP^{dc}(vk) = \left(E\left(\int\limits_{((v-1)k,vk]} \exp\left(\int\limits_s^{vk} r_u du\right) p^{dc}(s)\,ds | F_t \right)\right)^+, \tag{34}$$

where

$dP^{dc}(vk)$ is the future profit payment at time vk in a defined contribution plan
k is the periodicity, being $k = 1$ if annual, $k = 0.5$ if biannual, etc., and
$v \in \aleph - \{0\}$, $v = 1, 2, 3, 4, \ldots$.

In this type of plan, there is no modification in the calculation of the contractual provision. So Equation (29) must be rewritten as follows:

$$V_{be,bp}^{dc}(t) = E\left(\begin{array}{l} \left[\int\limits_{[t,n)} \exp\left(-\int\limits_t^s r_u du - \int\limits_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right) \right. \\ \quad \left. \times \left(-dp^1(s) + \mu_{x+s}^{(12)} V_c^{dc}(s)\,ds + \mu_{x+s}^{(13)} V_c^{dc}(s)\,ds + I_{(s=vk)} \cdot dP^{dc}(s)\right)\right] + \\ \left[\int\limits_n^{\infty} \exp\left(-\int\limits_t^s r_u du - \int\limits_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right) \left(db^1(s) + I_{(s=vk)} \cdot dP^{dc}(s)\right)\right] | F_t \end{array} \right), \tag{35}$$

where $V_{be,bp}^{dc}(t)$ is the best estimate of the economic technical provision of a bonus plan at time t

The corollaries presented in Section 3.1.1 can be extended to this section.

4.1.2. Dividend Plan

As can be seen in the defined benefit plan, with this type of profit participation the contractual provision depends not only on the first order parameters but also on the profits gained until the moment of calculation of the provision. To calculate the contractual provision as well as the best estimate, one needs to know the values of each of the past profit payments, which are evaluated using the following formula (Norberg 2001; Ramlau-Hansen 1991), considering the characteristics fixed in the annuity contract:

$$\begin{cases} dP^{o,dc}(vk) = \int\limits_{((v-1)k,vk]} \exp\left(\int\limits_s^{vk} r_u^o du\right) V_{c,dp}^{dc}(s-)\,\beta\left(r_s^o - r^*\right) ds & \text{if } vk < n \\ dP^{o,dc}(vk) = \int\limits_{((v-1)k,vk]} \exp\left(\int\limits_s^{vk} r_u^o du\right) V_{c,dp}^{dc}(s-)\,\beta\left(r_s^o - r^* + \mu_{x+s}^{o(12)} - \mu_{x+s}^{*(12)}\right) ds & \text{if } vk \geq n \end{cases}, \tag{36}$$

where

$dP^{o,dc}(vk)$ is the profit paid at time vkin a defined contribution plan with dividends
$vk < t$ and
$V_{c,dp}^{dc}(t)$ is the contractual technical provision at time t of a defined contribution plan with dividends.

As in Section 3.1.2, the units of each additional income purchased before the provision evaluation time is attained as following expression:

$$k^{o,dc}(s) = I_{(s=vk)} \frac{dP^{o,dc}(s)}{a(s)}, \tag{37}$$

where

$vk < t$
$a(s)$ is the contractual annuity (Eq. [19]) and
$k^{o,dc}(s)$ is the additional income purchased at time s in a contribution plan with dividends.

The accumulated amount of units of additional income purchased until time of evaluation is calculated using the following expression:

$$dK^{o,dc}(u) = \int_{(0,t)} k^{o,dc}(s)ds, \quad s < t \text{ and } u \geq n, \tag{38}$$

where $dK^{o,dc}(u)$ is the additional income that is paid at time u, which was acquired by profits paid before the time of evaluation.

As a result of the profit payments, Equation (28) has to be changed as follows:

$$\begin{cases} V_{c,dp}^{dc}(t) = \int_{[t,n)} \exp\left(-\int_t^s r^* du\right)\left(-dp^1(s)\right) + \int_n^\infty \exp\left(-\int_t^s r^* du - \int_n^s \mu_{x+u}^{*(12)} du\right)\left(db^1(s) + dK^{o,dc}(s)\right) & \text{if } t < n \\ V_{c,dp}^{dc}(t) = \int_t^\infty \exp\left(-\int_t^s r^* du - \int_t^s \mu_{x+u}^{*(12)} du\right)\left(db^1(s) + dK^{o,dc}(s)\right) & \text{if } t \geq n \end{cases}. \tag{39}$$

Under solvency principles, to evaluate the best estimate of the technical provision, it is necessary to take into account the cash out-flows related to the units of additional income that will be obtained by each future profit payment (Eq. [34]). These units are calculated by dividing the results of Equations (34) and (19):

$$k^{dc}(s) = I_{(s=vk)} \frac{dP^{dc}(s)}{a(s)}, \quad s \geq t, \tag{40}$$

where $k^{dc}(s)$ is the estimate of the additional income purchased at time s in a defined contribution plan with dividends.

Consequently, the accumulated amount of units of the future additional income is found as follows:

$$dK^{dc}(u) = \int_t^u k^{dc}(s)ds, \quad u \geq n, \tag{41}$$

where $dK^{dc}(u)$ is the estimate of additional income that will be paid at time u in a defined contribution plan with dividends.

As a result, the best estimate of the economic provision is obtained by the following formula:

$$V_{be,dp}^{dc}(t) = E\left(\begin{array}{l}\left[\int_{[t,n)} \exp\left(-\int_t^s r_u du - \int_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right)\left(-dp^1(s) + \mu_{x+s}^{(12)} V_{c,dp}^{dc}(s)\, ds + \mu_{x+s}^{(13)} V_{c,dp}^{dc}(s)\, ds\right)\right] + \\ \left[\int_n^\infty \exp\left(-\int_t^s r_u du - \int_t^s \left(\mu_{x+u}^{(12)} + \mu_{x+u}^{(13)}\right) du\right)\left(db^1(s) + dK^{o,dc}(s) + dK^{dc}(s)\right)\right] | F_t \end{array}\right), \tag{42}$$

where $V_{be,dp}^{dc}(t)$ is the best estimate of the economic technical provision of a defined contribution plan with dividends at time t. To calculate the risk margin and economic provision it is necessary to follow Equations (5) and (6).

The corollaries presented in Section 3.1.2 can be extended to this section.

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5. CONCLUSION

This article presents explicit expressions for evaluating the technical provisions of traditional annuity plans using a modern actuarial approach, considering modern solvency principles. We have considered the characteristics of the Brazilian market, including the financial guarantee and profit participation rules, with the hope of contributing to the technical development of this market. As Brazilian insurance supervision follows the solvency principles of the IAIS and its solvency regime is considered equivalent to the European solvency regime by the European Insurance and Occupational Pensions Authority (EIOPA), we believe that our theoretical approach could be used in markets that present annuity plans with similar guarantees.

In this article we have assumed that the best estimate of the economic provision is calculated by the probability weighted average of future cash flows taking into account the time value of money under real-world measures. So we have worked with conditional expectation to obtain that value. Moreover we have assumed that policyholder behavior is not optimal, but we also have illustrated a calculation of the provision assuming optimal policyholder behavior to show the differences between both assumptions.

Several relevant conclusions about the values of the different provisions have been discussed to illustrate the characteristics of each plan. It is important to highlight our concern about the volatility of the best estimate of the economic provision given that in the case of surrender or death, the latter only in defined contribution plans, the value withdrawn is different from such provisions. As one has the right to withdraw the contractual provision, the volatility of the best estimate of the economic provision is not the same as the one presented by a similar hypothetical market. Given that, in general, contractual provision is a deterministic function, the best estimate will present a lower volatility than a similar bond. Moreover, in an economic solvency valuation, the flows of assets and liabilities will be different at the time of surrender (or death, in a defined contribution plan). Thus, to perform a defined asset-liability management, those flows have to be measured.

Under the principles of solvency, insurers do not need to account for the contractual provisions. However, this provision should be calculated because it is the surrender value, in the case of total surrender before the granting of income, and the value of the benefit in case of death, before the granting of income in the defined contribution plans. Furthermore, since it is necessary to use the forecasted surrender rates to calculate the values of the provisions, the surrender/lapse capital risk should be carefully evaluated by Brazilian insurers to maintain their solvency, as in Solvency II. An extension of this article is possible making the same approach to unit-linked plans, considering its embedded options.

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